Linear Algebra II

Semestral Examination

Instructions: All questions carry ten marks. Vector spaces are assumed to be finite dimensional.

1. Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

2. Let $T : V \to V$ be a linear operator on vector space of dimension $n \geq 2$ over a field F having two linearly independent eigenvectors with same eigenvalue λ. Prove that λ is a multiple root of the characteristic polynomial of T. Is the converse of this statement true? Justify your answer.

3. Let A be a real diagonalisable matrix of order n with r distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_r$. Let
$$V_i = \{v \in \mathbb{R}^n \mid Av = \mu_i v\}.$$
If W is any A-invariant subspace of $\mathbb{R}^n$, then prove that
$$W = (W \cap V_1) \oplus (W \cap V_2) \oplus \cdots \oplus (W \cap V_r)$$

4. Let V be a real vector space. Let $<,>$ be a symmetric bilinear form on V. Prove that there exists an orthogonal basis of V with respect to which the matrix of $<,>$ is
$$\begin{bmatrix} I_p & & \\ & -I_m & \\ & & 0_r \end{bmatrix}$$
Further, prove that the numbers p, m and r are independent of the choice of the orthogonal basis of V with respect to $<,>$.

5. Let T be a normal operator on a Hermitian vector space V. Assume that v is an eigenvector of T with eigenvalue λ. Prove that v is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$.